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MAM1020S **Tutorial 2/3** **August 2017**

Note that for the exams the following rule will apply so better getting use to it already with the tutorials: no credit will be given to unjustified answers. Justify all your answers completely. (Or with a proof or with a counter example) unless mentioned differently. No step should be a mystery or bring a question. The grader cannot be expected to work his way through a sprawling mess of identities presented without a coherent narrative through line. If he can't make sense of it in finite time you could loose serious points. Coherent, readable exposition of your work is half the job in mathematics. You will loose serious points if your exposition is messy, incomplete, uses mathematical symbols not adapted...

1. (a) Consider the set of points $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $1 = (x-1)^2 + (y-1)^2$. Is this set of points define the graph of a function?
   (b) Consider the set of points $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $y = x^2$. Is this set of points define the graph of a function?

2. Find the domain of each of the following functions: that is all the real values for which the following function are defined:
   (a) $f(x) = \dfrac{4+\sqrt{2-3x}}{4-\sqrt{2+x^2}}$
   (b) $g(x) = 3\sqrt{\sin x - 1} + 2\sqrt{-x^3+1}$
   (c) $h(x) = \dfrac{x-5}{x^3-3x^2-x}$
   (d) $f(x) = \sqrt{2x^2-3x+4}$

3. We define the piecewise function $f : [0, +\infty[\to \mathbb{R}$

$$f(x) = \begin{cases} x, & if\ 0 \leq x \leq 1 \\ 2-x, & if\ 1 < x \leq 2 \\ 0, & if\ x > 2 \end{cases}$$

   (a) What is the domain of f? codomain of f?
   (b) Compute $f(0)$, $f(1/2)$, $f(1)$, $f(2)$, $f(3)$.
   (c) Compute the image of 5 by f.
   (d) Is the point $(1, 2)$ in $Graph(f)$?
   (e) Draw the graph. (Explain how you obtained it.)
   (f) Find all the preimages, if they exist, for $1, 0$ and 5
   (g) Is f one-to-one over $[0, +\infty[$? onto $\mathbb{R}$? bijective?
   (h) Compute the range of f.
   (i) Compute $f([0, 1/2])$ and $f(\{1, 2\})$.
   (j) Compute $f^{-1}([0, 1])$ and $f^{-1}(\{-2, -1, 0\})$.
   (k) Give the intervals where f is increasing? decreasing? constant?

4. Consider the functions

$$\begin{array}{rccc} f: & \mathbb{R} & \to & \mathbb{R} \\ & x & \to & -3x^2+4x+1 \end{array},$$

$$\begin{array}{rccc} g: & \mathbb{R} & \to & \mathbb{R} \\ & x & \to & -3x^2+7/3 \end{array}$$

and

$$\begin{array}{rccc} h: & \mathbb{R} & \to & \mathbb{R} \\ & x & \to & x-2/3 \end{array}$$

(a) Prove that for any $x \in \mathbb{R}$, $f(x) = -3(x - 2/3)^2 + 7/3$.

(b) Using the previous question find the roots f, that is all the values $x \in \mathbb{R}$ such that $f(x) = 0$.

(c) Compute the all the preimages of 5, -2.

(d) Compute $f(t^2)$, $f(s-1)$, $f(-x_1)$ and $f(x + 2/3)$.

(e) Give two points in $Graph(f)$.

(f) Prove that for any $x \in \mathbb{R}$, $f(x) = g \circ h(x)$. Remember $g \circ h(x) = g(h(x))$.

(g) Prove that h is increasing over $\mathbb{R}$.

(h) Prove that g is not increasing over $\mathbb{R}$. But that g is increasing over $(-\infty, 0]$ and decreasing over $[0, +\infty)$.

(i) Deduce that f is increasing over $(-\infty, 2/3]$ and decreasing over $[2/3, +\infty)$.

(j) Compute $f([0, 1])$, $f(\{1, 2, 3\})$.

(k) Compute $f^{-1}([0, 1])$, $f^{-1}(\{1, 2, 3\})$.

(l) Is f one-to-one? onto? bijective?

(m) Prove that g is even.

(n) Prove that f is neither odd nor even.

5. You have been employed by the receiver of revenue to improve their efficiency. The first thing they want from you is a formula that gives the tax paid on income earned. Here is the information they give you.

   - No tax is paid on any income that is less than or equal to R50 000.
   - For income greater than R50 000 and less than or equal to R100 000, you pay 10% on the excess above R50 000.
   - For income greater than R100 000 you pay the tax on R100 000 as defined above plus 20% on the excess above R100 000.

   Write down a (piecewise defined) function that gives the tax paid on any income. (The above figures are not correct, but illustrate the general idea about income tax.)
   Also draw a graph of tax paid against income earned.

6. (a) Solve $|-3x + 4| = 1$.

   (b) Solve $|-3x + 4| < |x^2 - 2|$.

   (c) Sketch the graph of $y = |-3x + 4| - |x^2 - 2|$.

7. Let $S : \mathbb{R} \to \mathbb{R}$ defined by $S(t) = 1$ if $0 \leq t < 1$ and $S(t) = 0$ for all other values of t.

   (a) Draw the graph of S: first draw a set of axes with the horizontal axis as the t-axis, then plot a few points. Finally decide what the entire graph looks like. (This is the graph that we are going to push around in this question; get the sketch of the graph checked.) What is the domain of S? What is its range?

   (b) Let $T : \mathbb{R} \to \mathbb{R}$ defined by $T(t) = S(t) - 1/2$. Draw the graph of T using the same set of axes used in the previous question. What have we really done to the graph of S to get the graph of T?

   (c) Let c be a constant, and $U : \mathbb{R} \to \mathbb{R}$ defined by $U(t) = S(t) + c$. Explain how the graph of U can be obtained from the graph of S. (If you're stuck, look again at the previous question; we have tried to generalise in this question what we did for a specific value of c in the previous question.)

   (d) Let $W : \mathbb{R} \to \mathbb{R}$ defined by $W(t) = S(t + 2)$. Draw the graph of S and W on a new set of axes. If you're stuck, use a table of values for t; try to see what we have done to S here to get W.

(e) Let $X : \mathbb{R} \to \mathbb{R}$ defined by $X(t) = S(t-2)$. Draw the graph of X in on the set of axes for your last question. (Why have we asked this question? What's the difference between this question and the last? Get your graphs for W and X checked.)

(f) Let a be a constant and $Y(t) = S(t+a)$. Give a rule for obtaining the graph of Y from S. (Hint: the two previous questions should help here.)

(g) Let $Z : \mathbb{R} \to \mathbb{R}$ defined by $Z(t) = S(S(t))$ (What does that mean? If in doubt, ask!) Draw the graph of Z on a new set of axes. (Use tables if you are stuck.)

(h) Let $A : [0,2) \to \mathbb{R}$ defined by $A(t) = S(t)$ for $0 \le t < 2$ and let $A(t) = A(t-2)$ for $2 \le t < 4$. ($S(t)$ is the same function used in the previous two questions.)

   i. Sketch the graph of A on a new set of axes. What is the domain of A?

   ii. Let $B : [0, +\infty) \to \mathbb{R}$ defined by $B(t) = S(t)$ for $0 \le t < 2$ and $B(t) = B(t-2)$ for all $t > 2$. Draw the graph of B on a new set of axes. (Do you have enough information about B? Try to calculate $B(3)$, $B(5)$; this may give you a hint.)

   iii. Suppose we let $C : \mathbb{R} \to \mathbb{R}$ defined by $C(t) = S(t)$ for $0 \le t < 2$, and $C(t) = C(t-2)$ for all $t \notin [0,2)$. Draw a graph of C. (By the way: if you think this is an incredibly stupid function to be thinking about, don't do electrical engineering!)

8. Describe (domain, codomain, rule) $g \circ f$ and $f \circ g$ for the following pairs of functions and decide if those are equals or not:

   (a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x$ and $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 2x^2 - 5$,

   (b) $f : \mathbb{R} \to \mathbb{R}^+$ defined by $f(x) = e^{4x}$ and $g : \mathbb{R}^+ \to \mathbb{R}$ defined by $g(x) = \sqrt{x}$,

9. Decompose the following functions into the form $g \circ f$ (describe fully $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, there is not only one answer possible so you might be right and still have a different result from your friend):

   (a) $6x + 3$,

   (b) $4x^2$

   (c) $4x^2$ (in a different way),

   (d) $e^x + 4$,

   (e) $x^2 + 2x + 1$.

10. Let

$$f : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{R} \\ x & \to & -3x + 10 \end{array},$$

   (a) Prove that f is onto.

   (b) Prove that f is one-to-one.

   (c) Is f bijective?

   (d) Compute the inverse of f.

   (e) Verify that the inverse computed in the previous question is indeed an inverse using the identity seen in class for inverses with composition.

11. Let

$$f : \begin{array}{ccc} [0, +\infty) & \to & [0, +\infty) \\ x & \to & x^2 \end{array},$$

   (a) Prove that f is onto.

   (b) Prove that f is one-to-one.

   (c) Is f bijective?

   (d) Compute the inverse of f.

(e) Verify that the inverse computed in the previous question is indeed an inverse using the identity seen in class for inverses with composition.

12. A certain bacteria colony is know to have a doubling time of 3 hours. Suppose that you are infected with 50 of these bacteria when you cut yourself on something. What is the size of the population after 15 hours? After t hours? Give the size of the population after 20 hours.

13. You sell hot chocolate at the waterfront, and you find that sales vary with the time of year. The lowest sales, of 50 litres a day, happen on 1 February. The highest sales, of 350 litres a day, happen on 1 August. We want to model this situation with a sine curve, where $t = 0$ is the start of January. You will see that this is actually an exercise in shifting and scaling the standard sine function.

    (a) Begin by drawing a graph that covers the period of a year, putting in the in- formation given above. Units for the horizontal axis should be months. (12 in a year?) We would now like a formula for the amount of hot chocolate sold at any time, using this sine curve model. Use the function
    $$f(t) = [asin(bt - c)] + d$$
    where t is time measured in months, and $f(t)$ is the number of litres of hot chocolate sold at time t. You need to find out what the correct constants a, b, c, d are, for this situation. Find them by following these steps:

    (b) What is the amplitude you want for your graph? Since the ordinary sine function has amplitude 1, you will need to stretch the sine function vertically to get the amplitude you want. How much do you need to stretch? Which constant have you just worked out?

    (c) If you take an ordinary (stretched) sine function, it oscillates about the x axis. You want your function to oscillate around what value? That means that you want to shift your function. This tells you another constant. Which one?

    (d) That leaves b and c to be found. Do that as follows: substitute in the co-ordinates of the points that correspond to 1 February and 1 August. That leaves you with two equations in two unknowns. Solve them simultaneously. (You will get many solutions; just pick one for b and one for (c).

    (e) Write out the final formula for $f(t)$.

14. Find $f \circ g$, $g \circ f$, $f \circ f$ and $g \circ g$ for f and g with rule $f(x) = \sqrt{x}$, $g(x) = \sqrt{1-x}$. What is the domain of definition of f and g (the maximal interval of $\mathbb{R}$ such that f and g are defined)? Give the domains over which the composite function above are defined. Describe some domain and codomain you could chose so that you can compose them. (several answers are possible) Compute those composition.

15. Express the following functions in the form $f \circ g$ (that is, give an f and g that will work):

    (a) $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) = \sqrt{1 + t^2}$,

    (b) $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) = cos^3(t)$,

    (c) $h : \mathbb{R} \to \mathbb{R}$ defined by $h(t) = \frac{cos(t)}{cos^2(t)+2}$.

16. Given that $g : \mathbb{R} \to \mathbb{R}$ defined by $g(x) = 2x + 1$ and $h : \mathbb{R} \to \mathbb{R}$ defined $h(x) = 4x^2 + 4x + 7$, find a function f such that $f \circ g = h$.